UDC 530.1

ON CONDITIONS AT SECOND-ORDER DISCONTINUITIES IN THE THEORY OF GRAVITATION

PMM Vol. 36, №1, 1972, pp. 3-14

L. I. SEDOV
(Moscow)

(Received September 16, 1971)

The construction of generalized models of a medium or field in the presence of internal degrees of freedom by using the fundamental variational equation is presented within the scope of the general theory of relativity. Conditions at second-order discontinuities are considered for components of the metric tensor.

A set of characteristic magnitudes with the related system of differential or, generally, functional equations for the determination of these is introduced in physical models of fields and continuous media. Some of the equations of this system are explicit expressions of known universal laws of conservation or are generalizations of these, while others are expressions of the kind of kinetic equations or of equations of state.

The construction of models and the statement of the problem involves, in addition to these equations, the formulation of conditions at second - order discontinuities within the region of a separated volume of medium and at its boundaries. The latter can, also, be formulated as conditions of a continuous or discontinuous contact of the given medium with separated external objects. In principle it is always possible to consider the conditions at the boundary as complete or simplified relationships at discontinuities in which the properties of a particular model and those of the model representing external media. Thus the conditions at second - order discontinuities or, in particular, those related to the continuity of contacts may be taken as the basis for establishing boundary conditions.

The methods of derivation of conditions at discontinuities within a medium or at its boundaries by integral representation of the laws of conservation and passing to limit from continuous processes in a given medium, or from such processes and phenomena in more complex media, to discontinuous processes in that medium are well known.

More sophisticated models whose volume elements are defined by a number of parameters, such as strain, mixture composition, structure of molecules and of their sets, dislocation properties, electromagnetic state, etc., are now being introduced in theoretical and practical applications.

The presence of these parameters results in the appearance of new "dynamic equations" with an increased number of conditions at first- and second-order discontinuities.

Interpretation of the physical meaning of these parameters is provided in certain cases by the formulation of macroscopic relationships which must be added to those already known in order to define the related parameters defining certain classes of phenomena. A more detailed examination would show that it is precisely in this manner that the fundamental physical properties of any physical macroscopic magnitudes is established.

Unified general regular methods related to fundamental physical theories were developed in [1-6] for constructing models by deriving a closed system of equations and conditions at discontinuities. These systems are equivalent to the single fundamental variational equation

$$\delta \int_{V_4} \Lambda d\tau + \delta W^* + \delta W = 0 \tag{1}$$

where Λ is the prescribed Lagrangian function whose arguments are the parameters and their derivatives with respect to time and coordinates which independently define the physical state of an "elementary particle" in four-dimensional elementary volumes $d\tau$ mentally isolated in any imaginary finite four-dimensional space-time volume V_4. The specified functional δW^* may contain a volume integral over V_4 and surface integrals over Σ, the boundary of V_4, and over the system of both sides of the discontinuity surface S_3 which may exist within V_4. The integrand of both integrals are linear functions of variations of the defining parameters. The presence of δW^* in Eq.(1) is associated with the interaction of the specified medium with external objects and also with the nonholonomic properties of the volume integral in δW^* due to the irreversible effects which may be also defined by Eq. (1).

Variations of defining parameters on Σ and S are in Eq.(1) non-zero. The additional term δW in Eq.(1) is there for the purpose of compensating the related surface integrals in δW^* and the integrals resulting from the variation of the first volume integral, whenever variations on the Σ-boundary are nonzero.

For the simplest conventional models of physical media the magnitude δW^* can be defined by the following formula containing only the volume integral:

$$\delta W^* = \int_{V_4} (\rho T \delta S - \mathbf{F} \delta \mathbf{r})\, d\tau \tag{2}$$

where ρ is the medium mass density, T is the temperature, S is the entropy of a unit of mass, and $\mathbf{F}$ and $\delta\mathbf{r}$ are four-dimensional vectors of the external ponderomotive force and the imaginary variation of shift at a given point, respectively. In three-dimensional treatment $\mathbf{F}\delta\mathbf{r}$ represents the work of a three-dimensional volume force and the related energy input.

If the viscosity properties of medium are taken into account, a surface integral defining the work of viscosity stresses at the Σ-surface appears in the expression for δW^* in addition to the to the volume integral.

The variational equation (1) integrated over volume V_4 represents in the general case the complete energy equation with all energy exchanges taken into account for the volume element $d\tau$ and extended to any infinitely small imaginary variation of defining parameters. The actual time-dependent variations of the latter are replaced in this generalized equation by imaginary variations.

Whenever among the defining parameters there are successive derivatives with respect to time, then unlike in the energy equation, additional terms may appear in the variational equation. In real processes these additional terms in the variational equation are exactly zero, while in processes subject to variations such terms may not vanish and can materially affect relationships derived from (1).

The relation of the fundamental equation to the conventional Lagrange variational principle and to the complete energy equation is defined by the underlying physical principles in the construction of the Lagrangian function Λ and of the functional δW^*. In particular, these data make it possible to apply methods and results of thermodynamics of irreversible processes to the determination of δW^* (the conventional and general principles of Onsager, various kinds of experimentally confirmed principles of entropy maximum or minimum or the increase of other magnitudes, the association law in the theory of plasticity, etc.).

As shown in the references cited above, the determination of the term δW for specified Λ and δW^* makes it possible to readily establish in the general case the equations of state for a medium (generalized equations of the kind of Hooke's law, of the laws of polarization and magnetization, etc.).

All of the foregoing general considerations and Eq.(1) can be applied in Newtonian mechanics, as well as within the scope of the special (STR) and the general (GTR) theory of relativity and of their further developments.

Let us consider as the basic example the fundamental equation (1), which is of intrinsic importance in the GTR, and find the related conditions at second-order discontinuities in a gravitational field. In connection with the considered theory we shall furthermore derive certain general notes which are of independent importance.

In the GTR a space-temporal continuum is represented by a four-dimensional Riemannian space whose metric in the observer's system of coordinates $x^1x^2x^3x^4$ is represented by the quadratic form

$$ds^2 = g_{ij}dx^idx^j \tag{3}$$

where g_{ij} $(x^1x^2x^3x^4)$ are components of the metric tensor.

At every point of space this form (3) can be locally reduced to the Gallilean form (an orthogonal system of coordinates)

$$ds^2 = c^2dt^2 - (dy^1)^2 - (dy^2)^2 - (dy^3)^2 \tag{4}$$

where c is the speed of light and dt is an increment of time.

The metric tensor components g_{ij} may be considered as the unknown parameters characterizing the internal degrees of freedom related to the properties of a physically defined Riemannian space. Since in the STR the space is a priori known, hence g_{ij} are to be assumed known and, consequently, can be selected in a pseudo-Euclidean space to a certain extent arbitrarily.

The symmetry of metric properties of a space is usually defined by the following statement.

The metric of a space is symmetric if there exists in it a nontrivial (transformations other than identical or a simple change of sign are considered here) group of transformations $x'^i = \varphi_A{}^i\,(x^1x^2x^3x^4)$ for which

$$g_{ij}'\,(x^k) = g_{ij}\,(x^k) = g_{pq}\,(x'^k)\;\frac{\partial\varphi_A{}^p}{\partial x^i}\,\frac{\partial\varphi_A{}^q}{\partial x^j}\;;\quad \{A\} \tag{5}$$

where $g_{ij}\,(x^k)$ and $g_{ij}'\,(x^k)$ are, respectively, the original and the transformed components of the metric tensor. Subscript A denotes here elements of a group of transformations.

The systems of coordinates which satisfy relationships (5) are completely equivalent as regards the metric properties of space.

Euclidean and pseudo-Euclidean spaces are symmetric, and in these an appropriate fixing of functions g_{ij} (x^h) of the metric tensor provides whole classes of equivalent coordinate systems, hence it is impossible to isolate in this way a unique system of coordinates. All coordinate systems of the related class, particularly of those classes in which the metric is defined by (4) and which are derived by the Lorentz transformations, are geometrically (kinematically) equivalent.

Since in the general case Riemannian spaces are nonsymmetric, the selection of appropriate functions g_{ij} (x^h) completely defines a unique system of coordinates; hence there are generally no metrically equivalent systems of coordinates in finite parts of Riemannian spaces of the general kind.

However, metric (4), locally introduced in a Riemannian space has in the pseudo-Euclidean space in contact with it all the properties of a pseudo-Euclidean space, and consequently is not uniquely defined.

Riemannian spaces of various particular forms can, obviously, have the properties of symmetry and a metric invariant with respect to various related groups of symmetry transformations.

In the STR the space is pseudo-Euclidean, and the class of equivalent inertial systems of coordinates conditionally introduced in it depending on their relationship with physical bodies isolated by special conditions.

In the GTR every global system of coordinates is geometrically and physically uniquely defined owing to the absence of symmetry.

Bt special local transformation of coordinates at every point of space it is, however, possible in the GTR to introduce in the contacting pseudo-Euclidean space the related metric (4) invariant with respect to Lorentz transformations. Owing to the invariant form of (4), such local transformations are not uniquely defined.

The fundamental principle of the Gallileo-Newton relativity is defined by the statement that all laws of nature, expressed in the form of relationships between magnitudes in various observer's systems of coordinates, retain their form in inertial systems of coordinates. In the GTR this principle of relativity is locally formulated at any point of a Riemannian space in local coordinates (4). The latter are local inertial coordinates analogous to the global inertial system of coordinates defined in the STR by physical bodies, e.g., the system of "fixed" stars.

In Newtonian mechanics, as well as in the STR and the GTR, further assumptions are made with respect to certain fundamental magnitudes and relationships, determined beforehand in global or local inertial systems of coordinates, as to the conservation of their scalar, vector, or tensor properties in any system of coordinates.

Numerous examples of specific magnitudes for which such assumptions are not valid. Hence it becomes necessary to introduce, as physical characteristics, magnitudes for which the stated above assumption ("the principle of covariance") is satisfied. When the principle of covariance is satisfied, the tensor form of various physical relationships in the various isolated, equivalent or, generally, arbitrary systems of coordinates remain unchanged, although the actually written relationships or their individual terms may vary in different systems of coordinates.

The actual determination of specific coordinate systems is made by using various conditions and constructions, and in a metric space by making certain admissible particular assumptions with respect to the form of functions g_{ij} $(x^1x^2x^3x^4)$ for the components of the metric tensor.

Since in a pseudo-Euclidean space the isolation of a specific system of coordinates is not possible on the basis of only geometric considerations, the actual definition of the coordinate system necessitates the establishment of the relation of the coordinate system to specific physical bodies.

The determination of the concomitant coordinate system necessitates the definition of specific points in a certain physical or, generally, mentally defined medium. Such points can be specified for physical media by three Lagrangian coordinates $\xi^1\xi^2\xi^3$ and a similar time-dependent coordinate ξ^4 drawn along the world line corresponding to the considered point. The concomitant systems of coordinates can, obviously, be chosen with a certain degree of arbitrariness which can be removed by supplementary conditions.

In the case of certain physically real or ideal media or bodies the specific observer's system of coordinates $x^1x^2x^3x^4$ can be introduced as the concomitant coordinate systems, which are extended on specific conditions over the whole space, and chosen as bodies of the reference frame.

In the nonsymmetric Riemannian space the observer's system of coordinates can be uniquely defined by a purely geometric construction, in particular by specifying functions g_{ij} $(x^1x^2x^3x^4)$ which is admissible for any Riemannian space.

It is, for example, possible to use synchronous coordinate systems in which a certain finite region of space containing point $t_0x_0{}^1x_0{}^2x_0{}^3$ the form of (3) is of the following special form:

$$ds^2 = c^2dt^2 + g_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha, \beta = 1, 2, 3) \tag{6}$$

A synchronous system of coordinates is uniquely defined, for components $g_{\alpha\beta}$ $(x^1x^2x^3,t)$ the following supplementary conditions are specified:

$$g_{11}(x^1x^2x^3,t_0) = -1, \quad g_{12}(x^1x^2x^3,t_0) = g_{13}(x^1x^2x^3,\ t_0) = 0$$

$$g_{22}(x_0{}^1x^2x^3,\ t_0) = -1, \quad g_{23}(x_0{}^1x^2x^3,\ t_0) = 0$$

with $g_{11} = g_{22} = g_{33} = -1$ at point $x_0{}^1,\ x_0{}^2,\ x_0{}^3,\ t_0$. The introduction of this system is based on the known Riemann theorem [7].

Let in a given nonsymmetric Riemannian space the observer's system of coordinates $x^1x^2x^3x^4$ be uniquely defined, and let there be a certain medium containing points defined by coordinates $\xi^1\xi^2\xi^3\xi^4$. We then have

$$ds^2 = g_{ij}(x^k)\,dx^idx^j = \hat{g}_{pq}(\xi^k)\,d\xi^pd\xi^q \tag{7}$$

The transformation of coordinates

$$x^i = x^i(\xi^1\xi^2\xi^3\xi^4) \tag{8}$$

is nothing else but the law of motion of the given medium in the observer's system of coordinates. In the general case the four functions (8) are uniquely defined by the following ten equations with partial derivatives:

$$g_{ij}(x^k)\frac{\partial x^i}{\partial\xi^p}\frac{\partial x^j}{\partial\xi^q} = \hat{g}_{pq}(\xi^k) \tag{9}$$

provided that functions $g_{ij}(x^k)$ and $g_{pq}(\xi^k)$ are known and, in accordance with [7] relate to the metric of one and the same Riemannian space.

Problems of mechanics can be reduced to finding functions $\hat{g}_{pq}(\xi^k)$ and, consequently, to the determination of all metric properties of the Riemannian space. Having determined these functions from the geometric conditions isolating the observer's system of coordinates x^k in a given space, we can find the transformation (8) and the components $g_{ij}(x^k)$.

Hence, if the physical equations determining $\hat{g}_{pq}(\xi^k)$ can be formulated independently of the observer's system of coordinates, i.e., as the equations defining relationships which are independent of the selection of the observer's system of coordinates x^i, the equations of motion defining the law of motion (8) in mechanics will follow from the equation for $\hat{g}_{pq}(\xi^k)$ and relationships (9) with the conditions for the selection of the observer's coordinate system taken into account. These concepts may be used for substantiating the known statement that the equations of momenta and energy are usually obtained as a corollary of Einstein's equations of the gravitational field. The preceding reasoning makes it possible to extend this deduction to more general models in the field theory.

However in the case of more general models the equations of the field theory may prove to be nonautonomous, i.e., they may contain characteristics of the law of motion (8). In such cases the properties of the physically or geometrically isolated observer's system of coordinates may prove to be essential, since it is in that system that the fundamental physical properties defining the laws of motion and the metric properties of space in particular are established. Hence in the case of more complex and, possibly, of simplified models (e.g., in the presence of the space-time metric symmetry) it is not possible to substitute equations of the law of motion for those of the field theory. This particular situation occurs in Newtonian mechanics and in the STR. In these theories the metric properties of space are simple and known, while the equations defining the law of motion of a medium are not simple consequences of the space metric properties, although the latter substantially affect the nature and form of these equations.

In the case of Riemannian spaces which have the properties of symmetry the solution of equations of motion which can be derived equations of the field theory, it is, obviously, necessary to use certain conditions of the kind of initial conditions, which may not be required for solving equations of the field theory in the concomitant system of coordinates.

In the construction of various models within the scope of the GTR we consider, in conformity with the principle of covariance, that Λ, δW^* and δW are four-dimensional magnitudes representing scalar functional dependent on the components of the four-dimensional tensors appearing in the arguments in the form of invariant scalar combinations.

Let us consider the variational equation (1) on the following particular assumptions.

As the set of determining parameters we take in the observer's system of coordinates x^i the following set of values:

$$g_{ij}(x^1, x^2, x^3, x^4), \qquad \frac{\partial g_{ij}}{\partial x^k} \qquad \frac{\partial^2 g_{ij}}{\partial x^k \partial x^e}, \qquad \mu^A(x^k)$$

$$\nabla_i \mu^A = \frac{\partial \mu^A}{\partial x^i} + F^{Aj}_{Bs}\Gamma^s_{ij}\mu^B, \qquad K_B(\xi^k)$$

and the functions of the law of motion at the following points of the medium:

$$x^i(\xi^1, \xi^2, \xi^3, \xi^4), \qquad \partial x^i / \partial \xi^j = x_j{}^i(x^1, x^2, x^3, x^4)$$

where μ^A are certain thermodynamic parameters, either scalars, or tensor components, defining the state of the medium (one of these can be the entropy or the temperature), and F_{Bs}^{Aj} denote the corresponding products of Kroneker symbols $\delta_q{}^p$ dependent on the construction of the superscript A of the μ^A-components. Variables ξ^1, ξ^2 and ξ^3 are the Lagrangian coordinates defining individual points of the metric space, ξ^4 is the time coordinate along the world line, and $K_B(\xi^k)$ are known predetermined functions (generalization of physical constants).

The values

$$g_{ij}(x^k), \qquad \mu^A(x^k), \qquad x^i(\xi^k)$$

are taken as the unknown functions. We define the variations of the introduced magnitudes by the equalities

$$\delta x^i = \tilde{x}^i(\xi^1, \xi^2, \xi^3, \xi^4) - x^i(\xi^1, \xi^2, \xi^3, \xi^4)$$

$$\delta\mu^A = \tilde{\mu}^A(\xi^1, \xi^2, \xi^3, \xi^4) - \mu^A(\xi^1, \xi^2, \xi^3, \xi^4) =$$
$$= \tilde{\mu}^A(\tilde{x}^i) - \tilde{\mu}^A(x^i) + \tilde{\mu}^A(x^i) - \mu^A(x^i) = \partial\mu^A + \delta x^k \nabla_k \mu^A$$
$$\delta g_{ij} = \partial g_{ij}; \qquad \partial \frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \partial g_{ij}}{\partial x^k}; \qquad \partial \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} = \frac{\partial^2 \partial g_{ij}}{\partial x^k \partial x^l}$$
$$\partial \nabla_i \mu^A = \nabla_i \partial \mu^A + F_{Bs}^{Aj} \mu^B \partial \Gamma_{ij}^s$$

The validity of equalities

$$\delta \frac{\partial x^i}{\partial \xi^j} = x_j{}^l \nabla_l \delta x^i - \delta x^l \nabla_l x_j{}^i, \qquad \text{where } \nabla_l x_j{}^i = \frac{\partial x_j{}^i}{\partial x^l} + x_j{}^s \Gamma_{sl}^i$$

where Γ_{sl}^i are Christoffel symbols, and of equalities

$$\delta d\tau = \left(\frac{\partial \sqrt{-g}}{\sqrt{-g}} + \nabla_l \delta x^l \right) d\tau$$
$$\left(g = |g_{ij}|, \qquad \frac{\partial \sqrt{-g}}{\sqrt{-g}} = \frac{1}{2} g^{ij} \partial g_{ij}\right)$$
$$\delta K_B = 0, \qquad \partial K_B = -\nabla_i K_B \delta x^i, \qquad \delta \Lambda = \partial \Lambda + \delta x^i \nabla_i \Lambda$$

can be readily ascertained.

Let us consider models with Λ and δW^* of the following form:

$$\Lambda = \Lambda(R, g_{ij}, \mu^A, \nabla_k \mu^A, x_j{}^i, K_B) \tag{10}$$

$$\delta W^* = -\int_{V_4} M_A \delta\mu^A d\tau \tag{11}$$

In this case for δW we obtain the formula

$$\delta W = \int_{\Sigma} \left(T^{ijk} \partial g_{ij} + G^{ijk} \frac{\partial \partial g_{ij}}{\partial s_n} + M_A{}^k \delta\mu^A + P_i{}^k \delta x^i \right) n_k d\sigma \tag{12}$$

where n_k are covariant components of the external unit vector normal to surface Σ bounding volume V_4. Tensors T^{ikj}, G^{ijk}, $M_A{}^k$ and $P_i{}^k$ are to be determined, and R is the scalar curvature of the Riemannian space

$$R = g^{il}R_{il} = g^{il}R_{isl}{}^{s}, \qquad R_{ijl}{}^{s} = \frac{\partial \Gamma_{il}{}^{s}}{\partial x^{j}} - \frac{\partial \Gamma_{lj}{}^{s}}{\partial x^{l}} + (\Gamma_{pj}{}^{s}\Gamma_{il}{}^{s} - \Gamma_{pi}{}^{s}\Gamma_{lj}{}^{p})$$

The following formulas:

$$\partial R = -R^{ij}\partial g_{ij} + \nabla_l W^l, \qquad \nabla_l W^l = g^{ij}\partial R_{ij}$$

$$W^l = (g^{ij}\delta_s{}^l - g^{il}\delta_s{}^j)\,\partial\Gamma_{ij}{}^s = (g^{ik}g^{lj} - g^{ij}g^{lk})\,\nabla_k\partial g_{ij}$$

are valid for the variations of R .

Using the derived variation formulas, we carry out variation in Eq. (1) and from the volume integral obtain the following Euler equations: (13)

for ∂g_{ij}

$$-R^{ij}\frac{\partial \Lambda}{\partial R} + \frac{1}{2}g^{ij}\Lambda + \frac{\partial \Lambda}{\partial g_{ij}} - \nabla_q B^{ijq} - (g^{ij}g^{qk} - g^{iq}g^{jk})\nabla_q\nabla_k\frac{\partial \Lambda}{\partial R} = 0$$

where $B^{ijq} = \frac{1}{2}(B_1^{ijq} + B_1^{jiq})$, and

$$B_1^{ijq} = \frac{1}{2}\left(\frac{\partial \Lambda}{\partial \nabla_i \mu^A}F_{Bs}^{Aq}g^{sj} - \frac{\partial \Lambda}{\partial \nabla_i \mu^A}F_{Bs}^{Aj}g^{sq} + \frac{\partial \Lambda}{\partial \nabla_q \mu^A}F_{Bs}^{Ai}g^{sj}\right)\mu^B$$

for δx^i

$$\nabla_s\left(\frac{\partial \Lambda}{\partial x_j{}^i}x_j{}^s\right) + \frac{\partial \Lambda}{\partial x_j{}^s}\nabla_i x_j{}^s + \frac{\partial \Lambda}{\partial K_B}\nabla_i K_B + M_A\nabla_i\mu^A = 0 \qquad (14)$$

for $\partial\mu^A$

$$\frac{\partial \Lambda}{\partial \mu^A} - \nabla_l\frac{\partial \Lambda}{\partial \nabla_l \mu^A} = M_A$$

Equation (13) is a generalization of equations of Einstein's GTR. Depending on the form of functions $\partial\Lambda/\partial R$, Eq. (13) may contain up to fourth-order derivatives of the metric tensor components.

After variation of Λ and integration by parts, for δW we obtain

$$\delta W = -\int\limits_{\Sigma+S^{\pm}}\left\{\left[B^{ijk} + (g^{ij}g^{kl} - g^{ik}g^{lj})\nabla_l\frac{\partial \Lambda}{\partial R}\right]\partial g_{ij} - \frac{\partial \Lambda}{\partial R}(g^{ij}g^{kl} - \right. \qquad (15)$$

$$\left. - g^{il}g^{kj})\nabla_l\partial g_{ij} + \frac{\partial \Lambda}{\partial \nabla_k \mu^A}\delta\mu^A + \left(\Lambda\delta_i{}^k + \frac{\partial \Lambda}{\partial x_s{}^i}x_s{}^k - \frac{\partial \Lambda}{\partial \nabla_k \mu^A}\nabla_i\mu^A\right)\delta x^i\right\}n_k d\sigma$$

where Σ is the three-dimensional boundary of V_4 and S is the three-dimensional surface of a second-order discontinuity within the V_4-volume. Integration over S is carried out over both sides of this surface.

According to the general theory [4, 6] the first two terms of the surface integral which contain variations ∂g_{ij} and $\nabla_l\partial g_{ij}$ can be reduced to the form of formula (12). To achieve this transformation in a simple manner we use a special system of coordinates in which at a given point of surface Σ or S the coordinate conditions specified below are satisfied.

The coordinate axis x' is directed along the vector of the normal to Σ or S (vector $\mathbf{n}$ at points of surface Σ or S is directed outward from volume V_4) ; the remaining coordinate lines at the given point are orthogonal to $\mathbf{n}$ and lie in a plane tangent to Σ or S. Here we limit our analysis to points at which the normal is anisotropic, i.e., $as_n \neq 0$ and the Σ - and S -surfaces are smooth. In such system of coordinates the quadratic form (3) is reduced at points of Σ or S to the form

$$ds^2 = g_{11}dx^{12} + g_{\alpha\beta}dx^\alpha dx^\beta \qquad (\alpha, \beta = 2, 3, 4) \tag{16}$$

Owing to the smoothness of the Σ - and S -surfaces at the considered points of these surfaces we have the following equalities:

$$\frac{\partial g_{1\alpha}}{\partial x^\beta} = \frac{\partial g^{1\alpha}}{\partial x^\beta} = 0 \qquad (\alpha, \beta = 2, 3, 4)$$

Let us examine the results of transformation of the integrand in formula (15) for a given point of surface Σ in the described above special system of coordinates. Owing to the arbitrariness of related variations and of the Σ -surface, after equating (12) and (15) we obtain

$$\begin{aligned} -T^{11k}n_k &= B^{11k}n_k + \frac{1}{2} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial s_n} g^{11} \frac{\partial \Lambda}{\partial R} \\ -T^{1\gamma k}n_k = -T^{\gamma 1k}n_k &= B^{1\gamma k}n_k - \frac{1}{\sqrt{g^1}} g^{\gamma\beta} \frac{\partial \left(\frac{\partial \Lambda}{\partial R} g^1 \right)}{\partial x^\beta} \\ -T^{\alpha\beta k}n_k &= B^{\alpha\beta k}n_k + g^{\alpha\beta} \frac{\partial\, (\partial\Lambda/\partial R)}{\partial s_n} - \frac{1}{2} \frac{\partial \Lambda}{\partial R} \frac{\partial g^{\alpha\beta}}{\partial s_n} \end{aligned} \tag{17}$$

Here and in the following α, β and $\gamma \approx 2,3,4$, and $dS_n{}^2 = g_{11}dx^{12}$ is the invariant interval (*) along the normal to Σ . For an area element of Σ we have $d\sigma = \sqrt{G}\, dx^2dx^3dx^4$, where

$$G = |\, g_{\alpha\beta} \,| = \frac{1}{|\, g^{\alpha\beta} \,|}$$

We further have

$$G^{11k}n_k = G^{\gamma 1k}n_k = G^{1\gamma k}n_k = 0, \qquad G^{\alpha\beta k}n_k = \frac{\partial \Lambda}{\partial R} g^{\alpha\beta} \tag{18}$$

Moreover, owing to the arbitrariness of $\delta\mu^A$, δx^i and Σ , and independently of the coordinate system selection, we have

$$-M_A{}^k = \partial\Lambda/\partial \nabla_k \mu^A \tag{19}$$

and

$$-P_i{}^k = \Lambda \delta_i{}^k + \frac{\partial \Lambda}{\partial x_s{}^i} x_s{}^k - \frac{\partial \Lambda}{\partial \nabla_k \mu^A} \nabla_i \mu^A \tag{20}$$

Formulas (17) - (20) can be considered as equations of state which together with the energy-momentum tensor $P_i{}^k$ – the four-dimensional generalization of the conventional three-dimensional internal stress tensor – define the new tensors $M_A{}^k$, T^{ijk} and G^{ijk} which determine internal interactions in a field and in a medium [2, 3].

In view of the arbitrariness of volume V_4 and owing to the assumed arbitrariness of functions x continuous on S and of variations ∂g_{ij}, $\partial\partial g_{ij}/\partial S_n$, $\delta\mu^A$ and δx^i, we obtain with the use of formulas (17) - (20) from the integral over the two sides of surface S the following conditions at a discontinuity at points of surface S_i:

---

*) In the general theory, when $g_{11} < 0$, the interval ds_n is purely imaginary. Since the products appearing in the integrands are always real, there is no need to use only real definitions for ds, n_k and $d\sigma$.

$$[T^{ijk} n_k \sqrt{G}]_1^2 = 0 \tag{21}$$

$$[G^{ijk} \overline{n_k} \sqrt{G}]_1^2 = 0 \tag{22}$$

$$[M_A{}^k n_k \sqrt{G}]_1^2 = 0 \tag{23}$$

$$[P_i{}^k n_k \sqrt{G}]_1^2 = 0 \tag{24}$$

The vector of normal n is directed here from side 2 to side 1, and indices 1 and 2 in the notation $[A]_1^2 = A_2 - A_1$ denote different side of surface S.

In the derivation of equalities (21) - (24) allowance is made for the continuity of coordinates x^i and of their derivatives on surface S, although the metric and the area elements $d\sigma = \sqrt{G} dx^2 dx^3 dx^4$ can be discontinuous.

The assumption of continuity of the x^i-coordinate and of variations ∂g_{ij}, $\partial\partial g_{ij}$ / $/\partial S_n$, $\delta\mu^A$ and δx^i at intersection of S is bound with the assumption of absence of internal discontinuities, such as cracks or dislocations within the medium, and with the essential requirement for the conditions at discontinuities to reduce to identities when there is no second-order discontinuity at the S-surface. Conditions (24) obviously coincide with the usual equations of momentum and energy conservation at a discontinuity. Conditions (23) apply only when function Λ depends on gradients $\nabla_k \mu^A$ and on certain parameters μ^A. The tensor properties and the related invariance of relationships (21) - (24) follow from the assumption of the continuity of differentials dx^α and on the scalar nature of $\sqrt{G} dx^2 dx^3 dx^4 = d\sigma$.

Relationships (21) and (22) imply the imposition at discontinuities of certain conditions on the components of the metric tensor. In the case of continuous transformation of coordinate systems the actual relationships derived from conditions (21) and (22) can change their form, if the transformed derivatives of coordinates with respect to original coordinates become discontinuous at intersections with the S-surface. The derivatives ∂x^i / $\partial\xi^\kappa$ are generally discontinuous at the S-surface, hence conditions (21) and (22) depend on the observer's system of coordinates at different sides of that surface.

The introduction of the special coordinates in which formulas (17) and (18) were derived is linked with the coordinate transformations with discontinuous derivatives along S

Conditions (21) essentially depend on the presence among the Λ-arguments of gradients of parameters μ^A. However the presence of these may have no effect whatsoever, if the expressions $B^{ijk} n_k$ are continuous at S.

In the GTR proposed by Einstein we have

$$\Lambda = \frac{1}{2\varkappa} R + \Lambda_m \tag{25}$$

where $\varkappa$ is the dimensionless gravitational constant and Λ_m is the Lagrange function for a medium and an electromagnetic field independent of R. If it is further assumed that parameters μ^A are absent or that the expressions

$$\sqrt{G} B^{ijk} n_k \tag{26}$$

are continuous, then the conditions at discontinuities (21) and (22) reduce to the very simple form

$$\left[\sqrt{G} g^{11} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial s_n}\right] = 0, \qquad \left[\sqrt{G} g^{\gamma\beta} \frac{\partial \sqrt{g^{11}}}{\partial x^\beta}\right] = 0 \tag{27}$$

$$\left[\sqrt{G}\,\frac{\partial g^{\alpha\beta}}{\partial s_n}\right] = 0, \qquad [\sqrt{G}g^{\alpha\beta}] = 0$$

It will be readily seen that these conditions together with the adopted system of coordinates are equivalent to the requirement for the absence of second-order discontinuities of all metric tensor components and of derivatives of components $g^{\alpha\beta}$ with respect to s_n This conclusion is, however, dependent on the specific form of formula (25) with constant coefficient $\varkappa$ and on the continuity condition (26). These properties may not necessarily exist in some of the new models.

Moreover, by virtue of (18) it follows from (22) that in the chosen system of coordinates the derived here conditions of continuity of components $g^{\alpha\beta}$ and $g_{\alpha\beta}$ remain valid at S provided $\partial\Lambda/\partial R$ is continuous at S and, in particular, that Λ is of the form (25).

The obtained continuity of the metric tensor components will be violated in other systems of coordinates derived from the introduced special system of coordinates by transformations with discontinuous derivatives at the S-surface.

BIBLIOGRAPHY

1. Sedov, L. I., Mathematical methods of constructing new models of continuous media. Uspekhi Matem. Nauk, Vol. 20, №5, 1965.
2. Sedov, L. I., On the energy-momentum tensor and macroscopic interactions in a gravitational field and in physical media. Dokl. Akad. Nauk SSSR, 1965.
3. Sedov, L. I., Variational methods of constructing models of continuous media, in: Irreversible Aspects of Continuum Mechanics and Transfer of Physical Properties in Moving Fluids. Springer Verlag, Wien-New York, 1968.
4. Sedov, L. I., Models of continuous media with internal degrees of freedom. PMM Vol. 32, №5, 1968.
5. Berdichevskii, V. L., Construction of models of continuous media by means of the variational principle. PMM, Vol. 30, №3, 1966.
6. Lur'e, M. V., Application of a variational principle in investigations of discontinuities in a continuum. PMM, Vol. 30, №4 1966.
7. Petrov, A. Z., Einstein's Spaces. Fizmatgiz, Moscow, 1961.

Translated by J. J. D.